1996 Boston Area Undergraduate Physics Competition

SOLUTIONS

1. (a) Just before B_1 hits the ground, both balls are moving downward with a velocity obtained via $mv^2/2 = mgh$; thus $v = \sqrt{2gh}$. Just after B_1 hits the ground, it moves upward with a speed v, while B_2 is still moving downward with speed v. The relative speed is therefore $2v$. Hence, after the balls bounce off each other, the relative speed is still $2v$ (as can be seen by working in the frame of reference where B_1 is motionless, and using $m_1 \gg m_2$). Since the speed of B_1 essentially stays equal to v, the upward speed of B_2 is therefore $2v + v = 3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$, or

$$H = d + 9h. \tag{1}$$

(b) Just before B_1 hits the ground, all of the balls are moving downward with a velocity obtained via $mv^2/2 = mgh$; thus $v = \sqrt{2gh}$.

Let us inductively determine the speed of each ball after it bounces off the one below it. If B_i achieves a speed of v_i after bouncing off B_{i-1}, then what is the speed of B_{i+1} after it bounces off B_i? Well, the relative speed of B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. The final upward speed of B_{i+1} is therefore $(v + v_i) + v_i$, so

$$v_{i+1} = 2v_i + v. \tag{2}$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), $v_3 = 7v$, $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{3}$$

which is easily seen to satisfy eq. (2), with the initial value $v_1 = v$.

¿From conservation of energy, B_n will bounce to a height of $H = l + (2^n - 1)^2 v^2/(2g)$, or

$$H = l + (2^n - 1)^2 h. \tag{4}$$

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming l is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

[Escape velocity from the earth is reached when $n = 14$. Of course, the elasticity assumption is absurd in this case, as is the notion that one may find 14 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{14}$.]

2. (a) It is fairly easy to see that at a point P far away from the ring the electric field must point slightly away from the axis (assuming that the ring is positively charged). This is because the distances from different points on the ring to P differ only by small second-order effects (on the order of the ratio of R^2 to the square of the distance from the ring to P), whereas the radial components of the different contributions to the field at P differ by first-order effects of this ratio.

Now, if we can demonstrate the fact that the field points inward toward the axis at some point on the cylinder, then we can simply invoke continuity of the field (since it points outward far away) to say that at some point the field lies along the cylinder. This demonstration can be done with a messy calculation for points in the plane of the ring, but it's easier to just use Gauss' Law: The total flux through the cylinder must be zero; therefore, given that it points outward somewhere (and that it obviously points outward at the ends of an imaginary cylindrical Gaussian surface), it must point inward somewhere.

(b) The reasoning is incorrect; the total force exerted by the bottom of B is *not* equal to the pressure times the area of the bottom. The bottom is exerting a force on the string (downward). Thus the pressure is actually a bit higher, since the sum of all the forces exerted by the bottom must indeed be the weight of the water.

[This is consistent with the fact that a little piece of area on the bottom of B has to support a taller column of water than one on the bottom of A. But just giving this column-of-water reasoning only says that the given reasoning is incorrect; it doesn't explain why it is incorrect.]

3. *First Solution:* The key is to realize that if a voltage difference is applied between points A and C, then points B, D, F, and H are all at the same potential (each one is, by symmetry, at a potential equal to the average of the potentials at A and C). Therefore, we can bring these four points together, without changing the resistance between A and C. We may then put the whole network in a plane, with A, C, G, and E as the vertices of a square, and one point (representing B, D, F, and H) in the middle. Each side of this square is a 1Ω resistor, and there are also four 1Ω resistors going from each of A, C, G, and E to the point in the middle. (The four remaining resistors out of the original twenty-four connect the identified points B, D, F, and H.)

We therefore equivalently have a square of 1Ω resistors, and a resistor of $\frac{1}{4}\Omega$ going from each vertex to a point in the middle. We may then identify the midpoint of AC with the point in the middle (since they are at the same potential); likewise for the midpoint of EG. This process essentially adds on a resistor of $\frac{1}{2}\Omega$ from each of A, C, G, and E to the center, while eliminating sides AC and EG of the square. So now we effectively have a 1Ω resistor between A and E, a 1Ω resistor between C and G, and $\frac{1}{6}\Omega$ resistors from each of A, C, G, and E to the middle. It is easy to see that this yields a resistance of $\frac{7}{24}\Omega$ between A and C.

Second Solution (due to Kiran Kedlaya): Let A be at a voltage of $+1$, and C be at a voltage of -1 (we may pick these to have magnitude 1, without loss of generality). Then by symmetry, the voltages of B, D, F, and H are all 0. Also, the voltages of E and G are $+x$ and $-x$, respectively, where x is to be determined.

Now, the current along any of the resistors is simply equal to the voltage difference of the endpoints (using $I = V/R$, with $R = 1\Omega$). So the sum of the currents going into, say, E (which come from the six points A, F, H, B, D, and G) is

$$I_{\to E} = (1-x) + (0-x) + (0-x) + (0-x) + (0-x) + (-x-x) = 1 - 7x. \tag{5}$$

But this sum must be 0. Hence $x = \frac{1}{7}$.

It is then easy to see, by the same reasoning, that the current flowing out of A (or into C) is $7 - x = \frac{48}{7}$.

Since the voltage difference between A and C is 2, and the current between them is $\frac{48}{7}$, the resistance between them must be $\frac{7}{24}\Omega$.

4. (a) i. Let us find the velocity (v_x, v_y) of the ball just after it bounces off the left ring. If the ball takes a time t to hit the other ring, then we must have

$$2v_y = gt. \tag{6}$$

Also, if the bounces on the rings occur at an angle of θ from the horizontal, then the horizontal distance between contact points is $2R(1-\cos\theta)$. Hence,

$$v_x t = 2R(1-\cos\theta). \tag{7}$$

Eliminating t from the previous two equations, and using $v_y = v_x \tan\theta$, gives

$$(v_x, v_y) = \sqrt{gR}\left(\sqrt{\cot\theta(1-\cos\theta)}, \sqrt{\tan\theta(1-\cos\theta)}\right). \tag{8}$$

Now, $\Delta P_x = 2mv_x$, so to maximize ΔP_x, we want to maximize $\cot\theta(1-\cos\theta)$. Setting the derivative of this equal to zero yields

$$\cos^3\theta - 2\cos\theta + 1 = 0. \tag{9}$$

An obvious root is $\cos\theta = 1$. The two other roots are then $\cos\theta = (-1 \pm \sqrt{5})/2$. The root of 1 is not the one we want, since $\Delta P_x \propto v_x \to 0$ as $\theta \to 0$ (since $1 - \cos\theta \approx \theta^2/2$, and $\cot\theta \approx 1/\theta$). So the maximum must occur at

$$\cos\theta = \frac{-1+\sqrt{5}}{2}. \tag{10}$$

(This happens to be the inverse of the golden ratio. θ is about $52°$.)

ii. A. Using eq. (8), along with $\tan\epsilon \approx \epsilon$ and $\cos\epsilon \approx 1 - \epsilon^2/2$ for small ϵ, we have:

- If $\theta = \epsilon$, with $\epsilon \to 0$, then

$$(v_x, v_y) = \sqrt{gR/2}\,(\epsilon^{1/2}, \epsilon^{3/2}). \tag{11}$$

To leading order in ϵ, the speed therefore goes like $S = \sqrt{gR\epsilon/2}$, which goes to zero, as $\epsilon \to 0$.

- If $\theta = \pi/2 - \epsilon$, with $\epsilon \to 0$, then

$$(v_x, v_y) = \sqrt{gR}\,(\epsilon^{1/2}, \epsilon^{-1/2}). \tag{12}$$

To leading order in ϵ, the speed therefore goes like $S = \sqrt{gR/\epsilon}$, which goes to infinity, as $\epsilon \to 0$.

B. Force is the change in momentum per time, so the average horizontal force, $\overline{F}_x$, needed to keep the rings together, is $\overline{F}_x = \Delta P_x/T$ where T the time between bounces. Since $\Delta P_x = 2mv_x$ and $T = 2v_y/g$, we see that $\overline{F}_x = \Delta P_x/T = mgv_x/v_y$. Hence,

$$\overline{F}_x = mg\cot\theta. \tag{13}$$

Therefore

- If $\theta = \epsilon$, with $\epsilon \to 0$, then

$$\overline{F}_x = mg\cot\epsilon \approx mg/\epsilon. \tag{14}$$

- If $\theta = \pi/2 - \epsilon$, with $\epsilon \to 0$, then

$$\overline{F}_x = mg\cot(\pi/2 - \epsilon) \approx mg\epsilon. \tag{15}$$

In the first limit, even though the speed of the ball approaches zero, the average horizontal force becomes infinite as $\theta \to 0$, because the bounces are so frequent. In the second limit, the speed of the ball approaches infinity, but the average horizontal force goes to zero as $\theta \to \pi/2$.
[Using the same reasoning, we see that the average vertical force is $F = \Delta P_x/T = mgv_y/v_y = mg$. This does not depend on the angle, which is the expected result; the rings are simply keeping the ball up above the ground, which requires an average force equal and opposite to the gravitational force, mg.]

(b) i. This question is equivalent to: for what $f(x)$ is v_x independent of x_0? If we look at a contact point on the right half, we see that

$$\frac{v_x}{v_y} = -f'(x_0). \tag{16}$$

The distance to the following bounce at $-x_0$ is $-2x_0 = v_x t = v_x(2v_y/g)$. Therefore,

$$v_x v_y = -g x_0. \tag{17}$$

Combining the previous two equations gives

$$v_x = -\sqrt{g x_0 f'(x_0)}. \tag{18}$$

For this to be independent of x_0 we must have

$$f(x) = a \ln(x) + b, \tag{19}$$

where a and b are arbitrary constants (with $a > 0$).

ii. As in part (ii) of part (a), we have $\overline{F}_x = mgv_x/v_y$. Using eq. (16), we obtain $\overline{F} = -mgf'(x_0)$. For this to be independent of x_0 we must have

$$f(x) = cx + d, \tag{20}$$

where c and d are arbitrary constants (with $c > 0$).

5. First, note that it is believable that the system will eventually come to halt, because it loses energy at each pivot point change. At each change, only the component of the mass's velocity that is perpendicular to the new pivot stick survives. The new speed is therefore $\cos\theta$ times the old speed. The energy thus decreases by a factor of $\cos^2\theta$ at each pivot point change.

   Let the angle between the pivot stick and the vertical be ϕ. The component of gravity acting on the mass in the tangential direction around the pivot point is $g\sin\phi \approx g\phi$, for small ϕ. $F = ma$ gives $\ddot{\phi} = (g/\ell)\phi$. The solution to this differential equation is (with $\gamma \equiv \sqrt{g/\ell}$)

$$\phi(t) = Ae^{\gamma t} + Be^{-\gamma t}, \tag{21}$$

   where A and B are constants determined by the initial conditions. If given initial conditions are $\phi(0) \equiv \phi_0$ and $\dot{\phi}(0) \equiv \omega_0$, then it is easy to solve for A and B, and we obtain

$$\phi(t) = \left(\frac{\gamma\phi_0 + \omega_0}{2\gamma}\right) e^{\gamma t} + \left(\frac{\gamma\phi_0 - \omega_0}{2\gamma}\right) e^{-\gamma t}. \tag{22}$$

   Note that $\dot{\phi}(t) = 0$ when

$$t = \frac{1}{2\gamma} \ln\left(\frac{\gamma\phi_0 - \omega_0}{\gamma\phi_0 + \omega_0}\right). \tag{23}$$

   We will find the total time of the movement by adding up the times between successive changes in pivot points. More specifically, if t_0 is the first time the pivot point changes, t_1 is the second time the pivot point changes, etc., then the total time is

$$T - t_0 = \sum_{n=1}^{\infty} t_n - t_{n-1}. \tag{24}$$

The quantity $t_n - t_{n-1}$ is given by twice the t in eq. (23) (because the mass must rise to the point where its speed is zero, and then fall back down), with $\phi_0 = \pm\theta/2$ and ω_0 equal to the angular velocity right after the nth change of pivot point. This angular velocity right after the nth change of pivot point is simply $\cos^n\theta$ times the angular velocity right before the first change in pivot point. The latter can be found by equating the kinetic energy right before the first change in pivot point to the loss in potential energy from the initial balancing point (and using $\cos\beta \approx 1 - \beta^2/2$); we obtain $\omega = \mp\gamma\theta/2$, just before t_0.

Therefore, the angular speed right after the nth pivot point change is equal to $\mp(\cos^n\theta)\gamma\theta/2$, and eq. (23) gives

$$t_n - t_{n-1} = \frac{1}{\gamma}\ln\left(\frac{1+\cos^n\theta}{1-\cos^n\theta}\right). \tag{25}$$

The total time, $T - t_0$, is thus

$$T - t_0 = \sum_{n=1}^{\infty}\frac{1}{\gamma}\ln\left(\frac{1+\cos^n\theta}{1-\cos^n\theta}\right). \tag{26}$$

Now let's try to simplify this, to show the leading behavior in $1/\theta$.

With $x \equiv \cos\theta$, we have, using the two hints,

$$\begin{aligned} T - t_0 &= \frac{1}{\gamma}\left(\sum_{n=1}^{\infty}\ln(1+x^n) - \sum_{n=1}^{\infty}\ln(1-x^n)\right) \\ &= \frac{2}{\gamma}\sum_{n=1}^{\infty}\sum_{k\ \mathrm{odd}}^{\infty}\frac{x^{nk}}{k} \\ &= \frac{2}{\gamma}\sum_{k\ \mathrm{odd}}^{\infty}\frac{1}{k}\frac{x^k}{1-x^k} \qquad \text{(doing the } n \text{ sum first).} \end{aligned} \tag{27}$$

We can now use the approximation $x^k = \cos^k\theta \approx (1-\theta^2/2)^k \approx 1 - k\theta^2/2$, but we must be careful to use this only in its range of validity, $k\theta^2/2 \ll 1$, i.e., $k \ll \theta^2/2$. We'll use the approximation for the range of k up to $M \equiv \eta/\theta^2$, with $\eta \ll 1$. In what follows, we will assume that θ is small enough so that we may choose η to satisfy $\theta \ll \eta \ll 1$. Note that we then have $M \gg 1$.

We may write $T - t_0$ as

$$T - t_0 \approx \frac{2}{\gamma}\sum_{k\ \mathrm{odd}}^{M}\frac{1}{k}\left(\frac{1-k\theta^2/2}{k\theta^2/2}\right) + \frac{2}{\gamma}\sum_{k\ \mathrm{odd},M}^{\infty}\frac{1}{k}\frac{x^k}{1-x^k}. \tag{28}$$

The second term is less than

$$\begin{aligned}
\frac{2}{\gamma}\sum_{k\ \mathrm{odd},M}^{\infty}\frac{x^k}{M(1-x^M)} &= \frac{2}{\gamma}\frac{1}{M(1-x^M)}\frac{x^M}{1-x^2} \\
&\approx \frac{2}{\gamma}\frac{1}{M(M\theta^2/2)}\frac{1}{\theta^2} \\
&= \frac{4}{\gamma\eta^2}. \qquad (29)
\end{aligned}$$

The first term is approximately equal to

$$\begin{aligned}
\frac{4}{\gamma\theta^2}\sum_{k\ \mathrm{odd}}^{M}\frac{1}{k^2} &\approx \frac{4}{\gamma\theta^2}\sum_{k\ \mathrm{odd}}^{\infty}\frac{1}{k^2} \\
&= \frac{\pi^2}{2\gamma\theta^2}. \qquad (30)
\end{aligned}$$

Since we have chosen η to satisfy $\theta \ll \eta \ll 1$, the first term above dominates, and we obtain

$$T - t_0 \approx \sqrt{\frac{\ell}{g}}\left(\frac{\pi^2}{2\theta^2}\right). \qquad (31)$$

6. (a) If the moment of inertia of C is zero, then there can be no torque on C (because otherwise there would be infinite angular acceleration). Hence the rubber band can apply no force to C. Therefore, C moves under the influence of only gravity, just like the block. So they have the same speed at all times (since they started out at the same speed).

   (b) i. Let's first define a few variables. These are all functions of t, but we won't bother writing the t dependence.

      A. Let f be the fraction of the distance C's position is along the rubber band.

      B. Let V_B be the speed of the block (then $V_B = tg\sin\theta \equiv g_\theta t$).

      C. Let v_b be the speed of the band at its point of contact with C. (then $v_b = fV_B = fg_\theta t$.

      D. Let V be the speed of C.

      E. Let ω be the angular velocity of C.

      F. Let a be the acceleration of C.

      G. Let α be the angular acceleration of C.

      H. Let F be the force the band applies to C, defined so that upward along the plane is positive.

Then, since $V = v_b + r\omega$, we have

$$V = fg_\theta t + r\omega. \tag{32}$$

Looking at the net force on C along the plane, we have $mg_\theta - F = ma$. Looking at the torque on C, we have $rF = \rho m r^2 \alpha$. Eliminating F gives $g_\theta - \rho r\alpha = a$. Integrating this from time zero to time t gives (since ω and V are zero at $t = 0$)

$$V = g_\theta t - \rho r\omega. \tag{33}$$

Adding eq. (33) plus ρ times eq. (32) gives

$$V = g_\theta t \left(\frac{1 + \rho f}{1 + \rho} \right). \tag{34}$$

So the ratio of C's speed to the block's speed is $(1 + \rho f)/(1 + \rho)$. Now, let f_∞ be the limiting value of f, as $t \to \infty$. Then for large t, ratio of C's speed to the block's speed is the constant $(1 + \rho f_\infty)/(1 + \rho)$. Therefore, for large t, the ratio of the distances travelled must be this same ratio, i.e.,

$$f_\infty = \frac{1 + \rho f_\infty}{1 + \rho}. \tag{35}$$

The solution to this equation is $f_\infty = 1$.

ii. Using the fact that the position of the block is $1 + g_\theta t^2/2$, we have, by definition of f,

$$\int V dt = f(1 + g_\theta t^2/2). \tag{36}$$

Differentiating this equation gives

$$V = fg_\theta t + (1 + g_\theta t^2/2)f'. \tag{37}$$

This says nothing more than that C's speed comes partly from the stretching of the band (the $fg_\theta t$ piece) and partly from the movement relative to the band (the term with f'). For large t, we may drop the f' term on the right, compared to the $g_\theta t^2 f'/2$ term. Then, using eq. (34), eq. (37) becomes

$$1 = f + \frac{1 + \rho}{2} f' t. \tag{38}$$

If we let the leading behavior of f be $f \approx 1 - ct^{-a}$, the previous equation gives $a = 2/(1 + \rho)$. Thus,

$$f \approx 1 - ct^{-2/(1+\rho)}. \tag{39}$$

The relative speed of the block and C is $V_{\rm rel} = V_B - V = g_\theta t - g_\theta t(1+\rho f)/(1+\rho)$. Using the form of f in eq. (39) gives

$$V_{\rm rel} = \frac{g_\theta \rho c}{1+\rho} t^{-\frac{1-\rho}{1+\rho}}. \tag{40}$$

A few examples:

A. If $\rho = 0$, then $V_{\rm rel} = 0$, as we found in part (a).

B. If $\rho = 1/2$, then $V_{\rm rel} \sim t^{-1/3}$, and goes to zero as $t \to \infty$.

C. If $\rho = 1$, then $V_{\rm rel}$ approaches a constant as $t \to \infty$.

D. If $\rho = 2$ (for example, a spool of thread that hangs down below the rubber band), then $V_{\rm rel} \sim t^{1/3}$, and goes to infinity as $t \to \infty$; but this relative speed becomes negligible compared to the block's speed $g_\theta t$, as $t \to \infty$; so f does indeed approach 1, as we found in part (i).